THEORETICAL COMPETITION

January 11, 2026

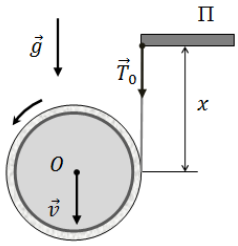
Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
3. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
4. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
5. Each task should be solved starting from a new page of the ***Writing sheets***
6. At the top of each sheet of paper your country (**Country**) and your student code (**Student Code**) are printed, you only need to fill in the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.

Problem 1 (10.0 points)

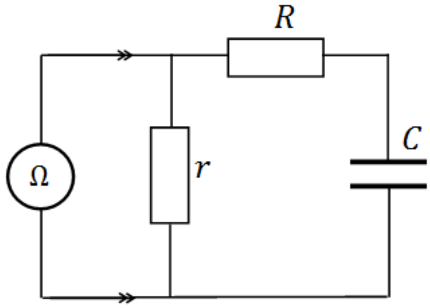
This problem consists of three independent parts.

Problem 1.1 (3.0 points)



A thin cylindrical core of radius R has a strip of foil tightly wound around it in a thin layer. The foil has thickness δ, mass m, and length $L = 10.0$ m. One end of the foil is fixed to the core, while the other end is attached to the edge of a thin horizontal platform Π. Initially, the roll (the core with the wound foil) is held so that its axis O is horizontal and at the same level as the platform. The roll is then released, and under the action of gravity it begins to unwind. Assuming that $\delta \ll R \ll L$, find the length x_0 of the unwound part of the foil for which the force T_0 acting on the platform is equal to the weight of the roll, mg. Also determine the acceleration a and the velocity v of the downward motion of the roll's center of mass at that instant.

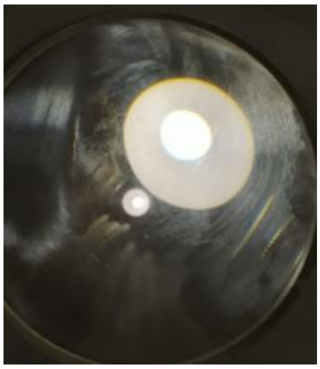
Problem 1.2 (4.0 points)



The readings of a multimeter (Ω) operating in ohmmeter mode and connected to a circuit change with time according to the law $R_\Omega(t) = A - Be^{-t/\tau_0}$, where $A = 100$ kΩ, $B = 40$ kΩ, and $\tau_0 = 100$ s. Using these data, determine the values of the resistances r and R, as well as the capacitance C of the capacitor, which is initially uncharged. Also find the amount of heat Q released in the resistor R during the measurement time $t \gg \tau_0$.

Note: In ohmmeter mode, the multimeter measures the voltage drop U_x across an unknown resistor R_x when a constant current $I_0 = 0.1$ mA flows through it. The instrument displays the value $R_x = U_x/I_0$.

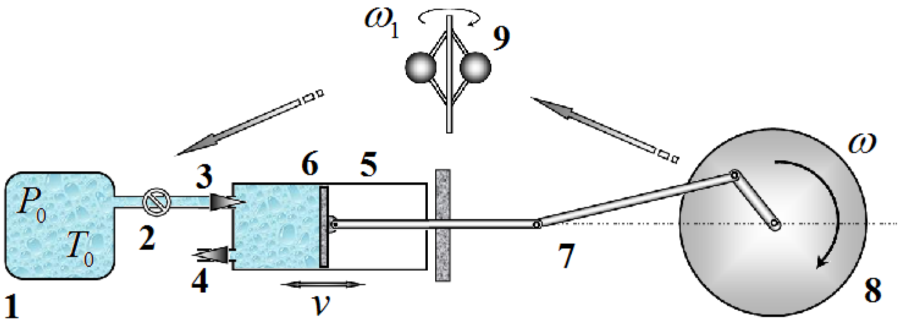
Problem 1.3 (3.0 points)



An observer (the eye) is located at a distance $L = 55$ cm from a convex–concave lens and observes images of a distant street lamp formed due to reflections from both surfaces of the lens (see photo). The lamp and the eye are positioned close to the principal optical axis of the lens on the side of the concave surface. The radius of curvature of the concave surface is $r = 7$ cm, and the radius of curvature of the convex surface is $R = 28$ cm. Upon observation, two images of the lamp are visible: one real and the other virtual. The ratio of the apparent angular sizes of these images is $\gamma = \varphi_1/\varphi_2 = 0.1$, where φ_1 is the angular size of the real image and φ_2 is the angular size of the virtual image. Find the optical power D of the lens.

Problem 2. Steam Engine (10.0 points)

In this problem, we consider a simplified theoretical model of one type of steam engine. Note that at present such an engine is used only as a technical toy.

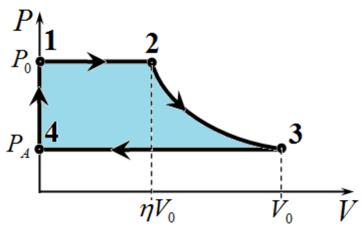


A schematic diagram of the engine under consideration is shown in the figure above. There is a steam generator 1 (a boiler with water and a heater, as well as a system for additional superheating of the steam). The generator may be regarded as a large vessel containing water vapor (without air), in which the pressure P_0 and temperature T_0 are maintained constant. The heated steam is supplied through a pipe to the working cylinder 5 with a movable piston 6. The pipe is equipped with a steam supply regulator 2 controlled by a Watt centrifugal governor 9. The working cylinder has two valves: an inlet valve 3 and an outlet valve 4. These valves open and close at certain positions of the piston. The movable piston is connected to a system of rods 7 that converts the translational motion of the piston into rotational motion of a massive flywheel 8 mounted on the engine shaft. By means of a belt drive, this wheel is connected to the Watt governor, whose angular velocity ω is equal to the angular velocity of the flywheel. The amount of steam passing through the steam supply regulator 2 depends on the rotational frequency of the governor. The flywheel shaft is connected to the working device (for whose sake all engines are created).

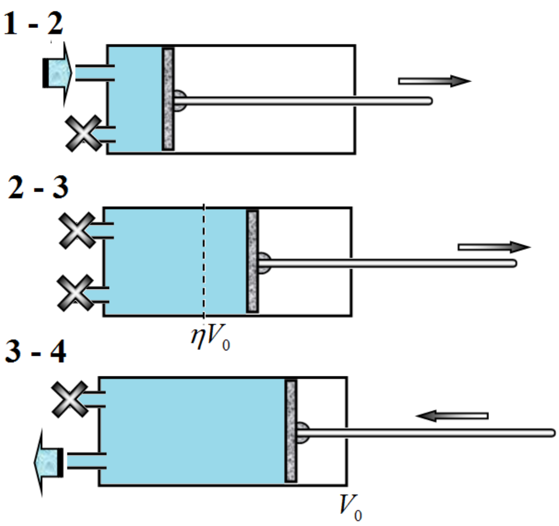
In all parts of the problem, steady operating regimes are considered, in which the angular velocities of rotation of the shaft and the Watt governor remain constant. Of course, the transition to such regimes is possible only in the presence of friction; however, in calculations of the steady motion, friction may be neglected.

Assume that the adiabatic index of water is $\gamma = 4/3$, the molar mass of water is $M = 18.0 \cdot 10^{-3}$ kg/mole.

Part 1. Steam Engine without a Governor



The ideal cycle of the engine under consideration in the limit of infinitely slow motion is shown on the $P-V$ diagram. Here V is the volume of the part of the cylinder between the cylinder end face and the piston that is occupied by steam (hereafter referred to as the working volume); $V_0 = 4.00$ L is the maximum working volume; and P is the pressure of the steam in this volume.



In state 1, the piston touches the cylinder wall and the working volume is zero. The inlet valve opens, and steam begins to enter the cylinder. Along segment $1-2$ of the cycle, the piston moves slowly to the right; in this process, the steam pressure in the cylinder may be assumed to be equal at all times to the pressure of the steam in the generator, $P_0 = 10.0 \cdot 10^5$ Pa.

At point 2, when the working volume reaches the value ηV_0, the inlet valve closes, and the piston continues to move to the right under the action of the steam pressure. Along segment $2-3$, the expansion process is adiabatic.

At point 3, the piston reaches its extreme position, the working volume is maximal, and the pressure drops to the atmospheric value $P_A = 1.00 \cdot 10^5$ Pa. After that, the outlet valve opens, and the piston expels the steam into the atmosphere.

When the piston returns to the extreme left position, the outlet valve closes and the inlet valve opens again (segment $4-1$).

2.1 Determine the value of the coefficient η.

To prevent steam condensation, its minimum temperature must be higher than the condensation temperature at atmospheric pressure, $t_S = 100°\text{C}$.

2.2 Determine the required value of the steam temperature T_0 (in degrees Celsius) such that during the expansion process the temperature does not drop below the condensation temperature t_S.

In what follows, assume that in the considered model the parameters T_0 and η are equal to the values found above.

2.3 Calculate the mass of steam m_0 entering the cylinder during one cycle.

2.4 Calculate the work A_0 performed by the engine during one cycle.

Next, consider the operation of the engine in a real regime, taking into account that the piston moves with some finite velocity. Assume that the flywheel rotates with a constant angular velocity ω. Strictly speaking, in this case the motion of the piston is not uniform. However, as an approximation, the piston velocity in one direction may be taken to be constant and equal to its average velocity. In this problem, it is convenient to consider the rate of change of the working volume, $v = dV/dt$.

2.5 Express the average rate of change of the working volume of the cylinder v in terms of the maximum working volume V_0 and the angular velocity ω of the flywheel.

During the piston motion, the steam pressure P in the cylinder during stage $1-2$ of the cycle differs from the steam pressure in the generator P_0. Assume that the rate of steam inflow is proportional to the pressure difference across the steam regulator 2, i.e.

$$\frac{dm}{dt} = K(P_0 - P),$$

where K is a constant coefficient determined by regulator 2. In this part, take it to be constant and equal to $K = K_0 = 4.20 \cdot 10^{-7}\ \text{kg/(Pa} \cdot \text{s)}$.

Assume that the steam expansion process at all stages of the cycle is adiabatic.

2.6 Derive a differential equation describing the change in the mass of gas in the cylinder while the inlet valve is open. Besides the unknown function $m(t)$, this equation should contain only known quantities.

2.7 Show that at a constant piston velocity the steam pressure P in the cylinder remains constant.

2.8 Obtain an algebraic equation for determining the steam pressure P in the cylinder during stage $1-2$ of the cycle.

To approximately solve the resulting equation (i.e., to find the pressure P), make the following mathematical approximation: assume (only in this part of the problem) that the adiabatic index of water vapor is $\gamma \approx 1$. In addition, when calculating the work performed by the steam, neglect atmospheric pressure.

2.9 Using the above approximation, express the steam pressure P in the cylinder during stage $1-2$ in terms of the pressure P_0 and the other known parameters of the problem.

2.10 Calculate the numerical value of the pressure P at an angular velocity of the shaft $\omega = 10.0\ \text{s}^{-1}$.

2.11 Show that the formula for calculating the work A performed by the engine during one cycle when the flywheel rotates with a constant angular velocity ω can be written in the form $A = \frac{A_0}{1+\beta\frac{\omega}{K}}$. Calculate the numerical values of the parameters A_0 and β in this formula.

Assume that, in addition to the torque M exerted by the piston, a constant torque M_0 from the working device acts on the engine shaft.

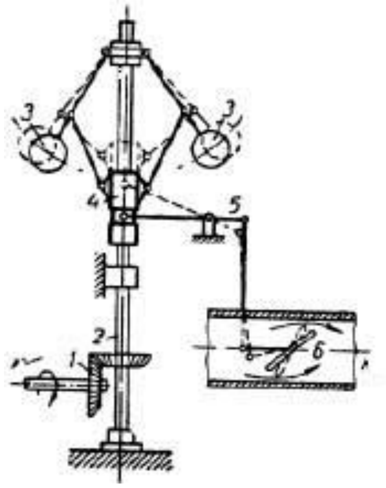
2.12 Express the average steady-state angular velocity ω of the wheel in terms of the torque M_0, the coefficient K, and the parameters A_0 and β.

2.13 Calculate the maximum torque M_0 at which the engine can operate.

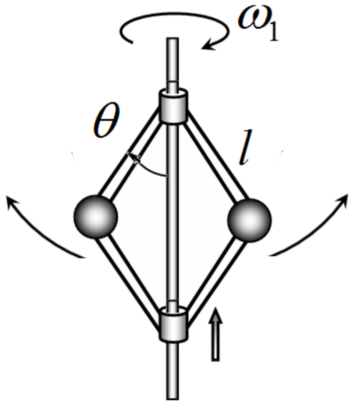
2.14 Sketch a schematic graph of the dependence of the angular velocity of rotation on the torque M_0.

Part 2. Governor without the Engine

The angular velocity of rotation of the engine flywheel depends on the torque transmitted to the working device. When this torque decreases, the angular velocity may increase sharply, which can lead to engine failure and its further destruction. To avoid such situations, various automatic control systems are used, one of which is the Watt centrifugal governor.



The principle of its operation is quite simple. The rotation of the engine shaft is transmitted through a drive to the shaft of the governor. During the rotation of the governor shaft, the balls deviate from the axis under the action of the centrifugal force; the faster the shaft rotates, the farther apart the balls move. The levers interact with a sleeve and move it along the shaft axis. Through a system of levers, the displacement of the sleeve is transmitted to the steam supply throttle in such a way that when the rotational speed of the shaft increases, the steam supply decreases, and when it decreases, the steam supply increases.



For solving this problem, the technical details of the specific device are not essential. Let us consider an idealized schematic model of the governor. A hinged rhombic frame is mounted on a vertical shaft rotating with a constant angular velocity ω. The upper vertex of the frame is fixed, while the lower vertex can freely slide along the shaft. Two massive balls are attached at the lateral vertices of the frame. It may be assumed that in the absence of shaft rotation $\theta = 0$. When the shaft rotates, the balls rise, deviating by an angle θ from the vertical. This leads to a reduction in the steam supply to the working cylinder. We assume that the coefficient K, which characterizes the steam flow rate, is related to the deflection angle by the simple relation

$$K = K_0 \cos\theta,$$

where K_0 is the coefficient defined earlier.

The length of a side of the frame is $l = 20$ cm.

2.15 Find the dependence of the deflection angle θ of the balls from the vertical on the constant angular velocity ω of the governor.

2.16 Sketch a schematic graph of the dependence of the coefficient K on the angular velocity ω of the governor.

Part 3. Engine with a Governor

2.17 Find the dependence of the steady-state angular velocity of rotation of the engine shaft on the torque M_0 for the governor system described above.

2.18 Sketch a schematic graph of the resulting dependence.

Problem 3. Electron Paramagnetic Resonance (10.0 points)

Magnetic Moment

The magnetic moment $\boldsymbol{m}$ of a planar current loop is a vector quantity defined as the product of the electric current I, the area of the loop S, and the unit normal vector $\mathbf{n}$ perpendicular to the plane of the loop:

$$\boldsymbol{m} = IS\boldsymbol{n}.$$

The direction of the magnetic moment vector is determined by the right-hand (corkscrew) rule: if the fingers of the right hand curl in the direction of the current in the loop, the thumb points in the direction of $\boldsymbol{m}$.

3.1 A circular loop of radius R carries an electric current and has a magnetic moment $\boldsymbol{m}$. Find the magnetic field induction $\boldsymbol{B}_0$ at the center of the loop.

3.2 The same loop is placed in an external uniform magnetic field of induction $\boldsymbol{B}$ such that the magnetic moment vector makes an angle φ with the direction of $\boldsymbol{B}$. Find the magnitude of the mechanical torque $\boldsymbol{M}$ acting on the loop due to the external magnetic field.

3.3 The loop is slowly rotated in an external uniform magnetic field $\boldsymbol{B}$ so that the direction of its magnetic moment $\boldsymbol{m}$ changes from being aligned with the field ($\boldsymbol{m} \uparrow\uparrow \boldsymbol{B}$)) to being opposite to the field ($\boldsymbol{m} \uparrow\downarrow \boldsymbol{B}$). Find the mechanical work A done by the magnetic field during this rotation.

Electron Paramagnetic Resonance

If a charged particle rotates or moves along a closed trajectory, it possesses a mechanical (angular) momentum, and as a result a magnetic moment also arises. In this case, a universal gyromagnetic relation holds, which shows how the magnetic moment and the mechanical moment of the particle are related.

Suppose an electron in an atom moves along a circular orbit such that its orbital angular momentum with respect to the center is $\boldsymbol{L}$. Such motion can be regarded as equivalent to a circular electric current, which has a magnetic moment $\boldsymbol{m}$. The magnetic moment and the angular momentum are related by the gyromagnetic relation

$$\boldsymbol{m} = -g_L \frac{e}{2m_e}\boldsymbol{L},$$

where e is the elementary charge, m_e denotes the electron mass, and g_L stands for the so-called Landé g-factor.

3.4 Find the Landé factor g_L for the circular orbital motion of an electron.

Electron Paramagnetic Resonance (EPR) is a phenomenon in which a substance containing unpaired electrons absorbs electromagnetic radiation (usually in the microwave range) when placed in a constant magnetic field. This absorption does not occur at arbitrary frequencies but only at a strictly defined one, and is therefore called resonant.

Assume that an atom has a single unpaired electron in its outer shell with zero orbital angular momentum. Such an electron has an intrinsic angular momentum called spin. When a sample of the substance is placed in a constant magnetic field produced by a solenoid with magnetic induction B, the spin can be oriented in two ways relative to the field: along the magnetic field direction with angular momentum projection $+\hbar/2$, or opposite to it with projection $-\hbar/2$. These two orientations have different energies, so transitions between them are possible. In what follows, assume that the Landé factor for the spin, g_s, is twice that for the electron's orbital motion. The sample is irradiated with an electromagnetic wave of fixed angular frequency ω, which can induce transitions of the electron between the two states with different spin projections. Then the magnitude of the external magnetic field induction is slowly varied, while the change in the absorption intensity of the electromagnetic radiation is recorded.

3.5 Find the angular frequency ω of the external electromagnetic radiation and calculate its numerical value if the maximum absorption occurs at a magnetic field induction $B_0 = 350$ mT.

Now atoms of the same type are embedded in an unknown material from which the core of the solenoid is made.

3.6 The magnetic field specified in the previous part was achieved at a current $I_0 = 1.50$ A in the solenoid winding. Determine the new current I in the solenoid winding required for resonant absorption if the magnetic permeability of the unknown material is $\mu = 1.25$.

Thermodynamic equilibrium

Assume that the core placed in the same magnetic field is in a state of thermodynamic equilibrium at a temperature $T = 50$ K, and the total number of embedded atoms is $N = 1.00 \times 10^{20}$. Let n_0 denote the difference between the number of atoms occupying the lower (N_1) and the upper (N_2) energy levels.

3.7 Calculate n_0 under the given conditions.

When electromagnetic radiation interacts with matter, three processes occur:

1) Absorption: an atom transitions from a lower energy level E_1 to a higher energy level E_2 by absorbing a photon. The number of transitions from the lower level per unit time is given by
$$\frac{dN_1}{dt} = -B_{12}\rho N_1,$$
where ρ is the energy density of the electromagnetic radiation;
2) Stimulated emission: under the influence of an external photon an atom transitions from a higher energy level E_2 to a lower energy level E_1 with the emission of another photon. The number of transitions from the upper level per unit time is given by
$$\frac{dN_2}{dt} = -B_{21}\rho N_2;$$
3) Spontaneous transition: a spontaneous transition of an atom from the upper level to the lower level accompanied by the emission of a photon. The number of such transitions from the upper level per unit time is given by
$$\frac{dN_2}{dt} = -A_{21}N_2.$$

The constants B_{12}, B_{21}, A_{21} are called Einstein's coefficients. Planck showed that in a state of thermodynamic equilibrium, the energy density of equilibrium electromagnetic radiation is described by the formula:
$$\rho = \frac{2\hbar\omega^3}{\pi c^3}\frac{1}{\exp\left(-\frac{\hbar\omega}{k_B T}\right)-1}.$$

3.8 Prove that $B_{12} = B_{21}$.

Presence of an external microwave field source

At the initial moment of time, the system is in thermodynamic equilibrium at the temperature specified above. Then a source of microwave radiation is switched on in such a way that the energy density of the electromagnetic radiation ρ in the sample remains constant in time, and its magnitude is such that spontaneous transitions in the system can be neglected.

3.9 Find the analytical time dependence $n(t)$ of the difference between the numbers of atoms occupying the lower and the upper energy levels, as a function of time t, assuming that $k = B_{12}\rho$.

3.10 It is known that the difference between the numbers of atoms occupying the lower and the upper energy levels changes by exactly a factor of 2 after a time $\tau = 1.00$ s from the moment the source is switched on. Under these conditions, calculate the power P of the microwave radiation source at the initial moment of time.

In reality, absorption and stimulated emission are not the only processes by which an electron in the upper energy level loses its excess energy. Relaxation processes play an important role, in which the excess energy is transferred to the surrounding matter; it is precisely due to these processes that an equilibrium distribution over energy levels is established.

The relaxation process for level 1 can be described as continuous transitions from level 1 to level 2 and back; the same is true for level 2. The terms describing relaxation for level 1 are written via constants α_1 and α_2 as
$$\frac{dN_1}{dt} = -\alpha_1 N_1 + \alpha_2 N_2,$$
and, consequently, a similar relation can be put down for level 2.

3.11 For the given substance, let $\alpha_1 + \alpha_2 = 0.670 \cdot 10^{-3}\ \mathrm{s}^{-1}$. Under these conditions, calculate the power of the microwave radiation source in the steady-state regime of spectrum measurement using the electron paramagnetic resonance method.

Mathematical hint for the theoretical competition

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$ is a fixed number, C refers to an arbitrary constant;

$\int \frac{dx}{x} = \ln|x| + C$, where C stands for an arbitrary constant;

$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$, where a – constant, C – arbitrary constant;

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, for $|x| \ll 1$ and any value of γ;

$\tan x \approx \sin x \approx x$, for $|x| \ll 1$;

$\ln(1+x) \approx x$, for $|x| \ll 1$.

List of physical constants

Acceleration of gravity $g = 9.80\ \mathrm{m/s^2}$;

Universal gas constant $R = 8.31$ J/(mole · K);

Speed of light in vacuum $c = 3.00 \cdot 10^8$ m/s;

Reduced Planck constant $\hbar = 1.05 \cdot 10^{-34}$ J·s;

Magnetic constant $\mu_0 = 4\pi \cdot 10^{-7}$ H/m;

Electric constant $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m;

Stefan-Boltzmann constant $\sigma = 5.67 \cdot 10^{-8}\ W/(m^2 \cdot K^4)$;

Elementary charge $e = 1.60 \cdot 10^{-19}$ C;

Electron mass $m_e = 9.11 \cdot 10^{-31}$ kg.